

# Sudden Expansion of a One-Dimensional Bose Gas from Power-Law Traps

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We analyze free expansion of a trapped one-dimensional Bose gas after a sudden release from the confining trap potential. By using the stationary phase and local density approximations, we show that the long-time asymptotic density profile and the momentum distribution of the gas are determined by the initial distribution of Bethe rapidities (quasimomenta) and hence can be obtained from the solutions to the Lieb-Liniger equations in the thermodynamic limit. For expansion from a harmonic trap, and in the limits of very weak and very strong interactions, we recover the self-similar scaling solutions known from the hydrodynamic approach. For all other power-law traps and arbitrary interaction strengths, the expansion is not self-similar and shows strong dependence of the density profile evolution on the trap anharmonicity. We also characterize dynamical fermionization of the expanding cloud in terms of correlation functions describing phase and density fluctuations.

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The vast majority of natural and laboratory-induced phenomena occur in interacting many-particle systems that are away from the equilibrium. Yet the nonequilibrium dynamics of such systems remains an unsolved problem both in quantum physics and some areas of classical physics such as fluid dynamics. Examples include quantum and classical turbulence, dynamics across phase transitions, and plasma instabilities, to name a few. Ultracold atomic gases have recently emerged as a particularly promising platform to gain new insights into aspects of nonequilibrium dynamics of quantum many-body systems [1, 2] and, more generally, into nonequilibrium statistical mechanics. In particular, there has been a surge of research activity in the study of the dynamics after a sudden quench of the system's parameters, exploring the mechanisms of relaxation and the role of integrability in approaches to equilibrium [3–11] (for further references, see [12, 13]).

In this Letter, we study far-from-equilibrium behavior of a trapped quantum gas after a sudden quench of the confining potential. More specifically, we investigate free expansion of an interacting one-dimensional (1D) Bose gas instantaneously released from the confining trap potential  $V(x)$ . This is a paradigmatic example of a “quantum explosion” problem in an experimentally realizable system that can be described by an integrable microscopic model—the Lieb-Liniger model [14] of delta-interacting bosons in one dimension. The exact integrability of the model offers an opportunity to investigate the expansion dynamics using theoretical methods that would have otherwise been inapplicable. At the same time, integrability implies that the underlying system lacks any mechanism of thermalization, which in turn poses a question of applicability of the standard hydrodynamic approach that was previously used to describe the dynamics of this system [15–19]. Combining these aspects together gives us a unique opportunity to: (i) solve the quantum explosion problem in a nontrivial manner, and (ii) benchmark the predictions of the hydrodynamic approach against those obtained here.

In this work, we treat the simplest case of expansion from

the zero-temperature ground state of the trapped gas. As we show below, the asymptotic density and momentum distributions of the gas after a sufficiently long expansion time (once the expansion becomes ballistic) can be obtained from the initial distribution of quasimomenta of the trapped (nonuniform) gas using the stationary phase approximation. This promotes the initial quasimomentum distribution from an auxiliary quantity—which has so far only been used to derive thermodynamic quantities—to the status of an observable physical property. The initial quasimomentum distribution itself is calculated by combining the exact solutions of the uniform Lieb-Liniger model and the local density approximation.

To start, we consider Hamiltonian evolution in free space of the many-body wave function, which—immediately prior to the removal of the trap potential at  $t = 0$ —describes the ground state of  $N$  trapped particles and is expanded in terms of the eigenfunctions of the uniform Lieb-Liniger model,

$$\Psi(x_1, \dots, x_N; t) = \frac{1}{(2\pi)^{N/2}} \int dk_1 \dots dk_N \times b(k_1, \dots, k_N) e^{i\Theta(k_1, \dots, k_N)} e^{i \sum_j (k_j x_j - \hbar k_j^2 t / 2m)}. \quad (1)$$

Here,  $b(k_1, \dots, k_N)$  are the expansion coefficients of the initial wave function [20], which depend on  $N$  different quasimomenta  $\{k_j\}$  (also referred to as rapidities, and having units of wave numbers) and are normalized to  $\int dk_1 \dots dk_N |b(k_1, \dots, k_N)|^2 = 1$ . The phase  $\Theta(k_1, \dots, k_N) = \sum_{j < l} \tan^{-1}[\hbar^2(k_l - k_j)/mg]$  arises from two-body collisions by the diffractionless delta-function interaction potential with the strength  $g$  [21].

The expansion coefficients  $b(k_1, \dots, k_N)$  determine the joint  $N$ -particle probability distribution of quasimomenta  $|b(k_1, \dots, k_N)|^2$ . In the single-particle sector, i.e., after integration over all quasimomenta but one, they give the quasimomentum distribution of the trapped gas,

$$g(k) = N \int dk_2 \dots dk_N |b(k, k_2, \dots, k_N)|^2, \quad (2)$$

with the normalization  $\int g(k) dk = N$ .

To proceed, we note that the only *time-dependent* term in the integrand of Eq. (1) is the phase of the last exponential. This exponential will, for sufficiently long times, develop fast oscillations as functions of  $k_j$  compared to the remaining *time-independent* terms. Therefore, in the long-time and large-distance limit, the asymptotic form of the wave function can be simplified significantly by using the stationary phase approximation [22]: the main contribution to the integral in Eq. (1) comes from the stationary phase points [20] satisfying

$$\frac{\hbar k_j}{m} = \frac{x_j}{t}, \quad (3)$$

and leading to the following asymptotic wave function:

$$\Psi_\infty(x_1, \dots, x_N; t) = \left(\frac{m}{\hbar t}\right)^{N/2} b\left(\frac{mx_1}{\hbar t}, \dots, \frac{mx_N}{\hbar t}\right) \times e^{i\Theta\left(\frac{mx_1}{\hbar t}, \dots, \frac{mx_N}{\hbar t}\right) + i\frac{m}{2\hbar t} \sum_j x_j^2 - i\pi N/4}. \quad (4)$$

The corresponding asymptotic density distribution  $\rho_\infty(x, t) = N \int dx_2 \dots dx_N |\Psi_\infty(x, x_2, \dots, x_N; t)|^2$  can therefore be found, using Eq. (2), as

$$\rho_\infty(x, t) = \frac{m}{\hbar t} g\left(\frac{mx}{\hbar t}\right), \quad (5)$$

with  $\int \rho_\infty(x, t) dx = N$ . Thus, the density profile after long expansion time is determined by the rescaled shape of the initial quasimomentum distribution  $g(k)$ ; finding this distribution constitutes, therefore, a key task of the present work.

The asymptotic wave function (4) also determines the asymptotic momentum distribution of the gas. Indeed, the Fourier transform  $\tilde{\Psi}_\infty(k_1, \dots, k_N; t)$  of Eq. (4) is again dominated by the stationary phase points satisfying Eq. (3), regarded now as conditions on the positions  $x_j$ . The result is

$$\tilde{\Psi}_\infty(k_1, \dots, k_N; t) = (-i)^N b(k_1, \dots, k_N) e^{i\Theta(k_1, \dots, k_N)}. \quad (6)$$

Integrating  $|\tilde{\Psi}_\infty(k_1, \dots, k_N; t)|^2$  over all momenta but one and using Eq. (2), one obtains the asymptotic momentum distribution,

$$n_\infty(k, t) = g(k), \quad (7)$$

implying that the initial quasimomenta of the trapped gas are mapped to real momenta of the expanded cloud [9, 22]. Then, the result of Eq. (5) for the density profile can simply be viewed as a consequence of the ballistic position-momentum correlations, Eq. (3), established in the long-time asymptotic regime, after the interaction energy has converted into the kinetic energy of expanding particles.

The characteristic expansion time  $t_e$  ensuring the applicability of the stationary phase approximation can be estimated by requiring that the fastest particles, moving with the velocity  $v_{\max} \sim c_0$ , where  $c_0$  is the speed of sound, overtake all slower particles. This is equivalent to  $c_0 t_e$  becoming larger than the characteristic size  $R$  of the initial cloud, and hence  $t_e \sim R/c_0$ . For expansion from a harmonic trap with frequency  $\omega_0$  this yields  $t_e \sim 1/\omega_0$  in the Thomas-Fermi approximation, in both the weakly and strongly interacting regimes (see below).

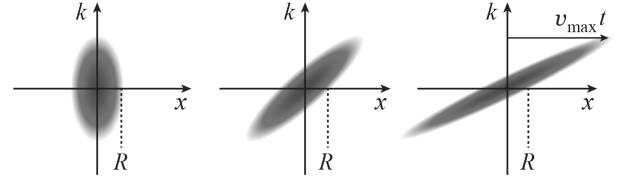


FIG. 1. Illustration of the evolving local quasimomentum distribution  $f(k, x; t) = f(k, x - \hbar kt/m; 0)$ , where  $v_{\max} = \hbar k_{\max}/m$ .

Finding the coefficients  $b(k_1, k_2, \dots, k_N)$  and hence the distribution  $g(k)$ , Eq. (2), is equivalent to solving for the ground state of the (nonintegrable) trapped gas and thus constitutes a formidable task for systems with large  $N$  [23]. However, as we argue here, for large  $N$ , the quasimomentum distribution  $g(k)$  can be found approximately within the local density approximation (LDA) [24].

The LDA is invoked by assuming that the initial trapped cloud can be divided into subsystems of length  $\Delta L$  small compared its overall size  $R$ , so that the density  $\rho(x) \equiv \rho(x, t=0)$  within each subsystem centered at  $x$  is approximately constant. At the same time  $\Delta L$  has to be sufficiently large compared to the microscopic correlation length  $\xi_0$  so that the locally uniform subsystem can be treated via the solution of the Lieb-Liniger integral equation [14] in the thermodynamic limit. Detailed conditions for the applicability of the LDA to trapped 1D Bose gases have been discussed in Ref. [25], in addition to being verified experimentally [26]. Generally speaking, the LDA is expected to be very good in the bulk of the atomic cloud for sufficiently large  $R$  (much larger than the length scale associated with the trapping potential), breaking down only in the small vicinity ( $\sim \xi_0$ ) of the cloud edge.

The solution to the Lieb-Liniger integral equation for each region gives the local quasimomentum distribution  $f(k, x; t=0)$  [14] corresponding to density  $\rho(x)$ , obtained for the local value of the chemical potential  $\mu(x) = \mu_0 - V(x)$  [25], where  $\mu_0$  is the global chemical potential. Integrating  $f(k, x; t=0)$  over  $x$  will give the quasimomentum distribution of the trapped gas in the local density approximation,

$$g(k) = \int f(k, x; 0) dx, \quad (8)$$

whereas  $\rho(x) = \int f(k, x; 0) dk$ , with the normalization condition  $\int f(k, x; 0) dk dx = N$ .

It is insightful to see how the asymptotic density evolution Eq. (5) can be obtained directly from the local quasimomentum distribution  $f(k, x; t)$  if the latter is provided with a semiclassical time dependence of the form  $f(k, x; t) = f(k, x - \hbar kt/m; 0)$ , for  $t \gg t_e$ . This choice makes use of the ballistic expansion relationship  $x = \hbar kt/m$  of Eq. (3) and is illustrated in Fig. 1. It is clear that such an evolution leaves the (quasi)momentum distribution intact,

$$g(k, t) = \int f(k, x - \hbar kt/m; 0) dx = g(k), \quad (9)$$

whereas the density distribution evolves according to

$$\rho(x, t) = \int f(k, x; t) dk = \int f(k, x - \hbar kt/m; 0) dk. \quad (10)$$

Introducing a new variable  $y \equiv x - \hbar kt/m$ , Eq. (10) can be rewritten as  $\rho(x, t) = \frac{m}{\hbar t} \int f\left(\frac{mx}{\hbar t} - \frac{my}{\hbar t}, y; 0\right) dy$ , where we can further neglect  $my/\hbar t$  in the first argument as the main contribution to the integral comes from the values of  $y$  that are of the order of the initial size of the cloud  $R$  and values of  $x \propto t$  that are much larger than  $R$  in the long-time limit. Using Eq. (8), one then obtains the same result as in Eq. (5),  $\rho_\infty(x, t) = \frac{m}{\hbar t} \int dy f\left(\frac{mx}{\hbar t}, y; 0\right) = \frac{m}{\hbar t} g\left(\frac{mx}{\hbar t}\right)$ , as anticipated.

We now apply our approach to expansion from power-law traps,  $V(x) = \frac{1}{2}\alpha_\nu |x|^\nu$ , where  $\nu \geq 2$  and  $\alpha_\nu$  is the confinement strength (in the important case of a harmonic trap,  $\nu=2$  and  $\alpha_2 = m\omega_0^2$ ). The problem can be treated analytically in two limiting cases: a weakly interacting gas ( $\gamma_0 \ll 1$ ) and a strongly interacting gas in the Tonks-Girardeau (TG) regime ( $\gamma_0 \rightarrow \infty$ ), where  $\gamma_0 = mg/(\hbar^2 \rho_0)$  is the dimensionless interaction strength in the trap center, with  $\rho_0$  being the peak density of the initial trapped sample. The intermediate regime can be addressed by finding numerically the local quasimomentum distribution  $f(k, x; 0)$  via the solution of the Lieb-Liniger integral equation [14] and then using Eqs. (8) and (5). The results of such a numerical treatment for a harmonic trap are shown in Fig. 2 (a), whereas the analytic results (see below) for  $\nu \neq 2$  are illustrated in Figs. 2(b) and 2(c).

In the weakly interacting regime ( $\gamma_0 \ll 1$ ), the local semiclassical distribution of a trapped gas with the density  $\rho(x)$  is given by [14]

$$f(k, x; 0) = \frac{1}{2\pi} \sqrt{\frac{4\hbar^2 \rho(x)}{mg} - \frac{\hbar^4 k^2}{m^2 g^2}}, \text{ for } |k| < K(x), \quad (11)$$

and  $f(k, x; 0) = 0$  otherwise, with  $K(x) = \sqrt{4mg\rho(x)/\hbar^2}$ .

The equation of state for a uniform gas in this regime is  $\rho = \mu/g$ , and therefore the density profile of the trapped sample in the Thomas-Fermi limit is given by

$$\rho(x) = \mu(x)/g = \rho_0 (1 - |x|^\nu/R^\nu), \text{ for } |x| < R, \quad (12)$$

and  $\rho(x) = 0$  otherwise. Here,  $R = (2\mu_0/\alpha_\nu)^{1/\nu}$  is the Thomas-Fermi radius, and  $\rho_0 = \mu_0/g = [(1 + \nu)^\nu \alpha_\nu N^\nu / (2^{\nu+1} \nu^\nu g)]^{1/(1+\nu)}$  is the peak density found from the normalization condition  $N = \int \rho(x) dx = 2R\rho_0\nu/(1 + \nu)$ .

Integrating the distribution function  $f(k, x; 0)$ , Eq. (11), over position gives

$$g(k) = \frac{RI_\nu}{\pi} \left( \frac{4\hbar^2 \rho_0}{mg} \right)^{1/2} \left( 1 - \frac{\hbar^2 k^2}{4mg\rho_0} \right)^{1/2+1/\nu}, \quad (13)$$

for  $|k| < \sqrt{4mg\rho_0/\hbar^2}$ , and  $g(k) = 0$  otherwise. Here,  $I_\nu \equiv \int_0^1 dy (1 - y^\nu)^{1/2} = \frac{\sqrt{\pi} \Gamma(1/\nu)}{(2+\nu) \Gamma(1/2+1/\nu)}$ , with  $\Gamma(z)$  being the gamma function. The asymptotic density distribution, from Eq. (5), is then determined by

$$\rho_\infty(x, t) = \frac{4I_\nu}{\pi} \frac{\rho_0}{\lambda(t)} \left( 1 - \frac{x^2}{\lambda(t)^2 R^2} \right)^{1/2+1/\nu}, \quad (14)$$

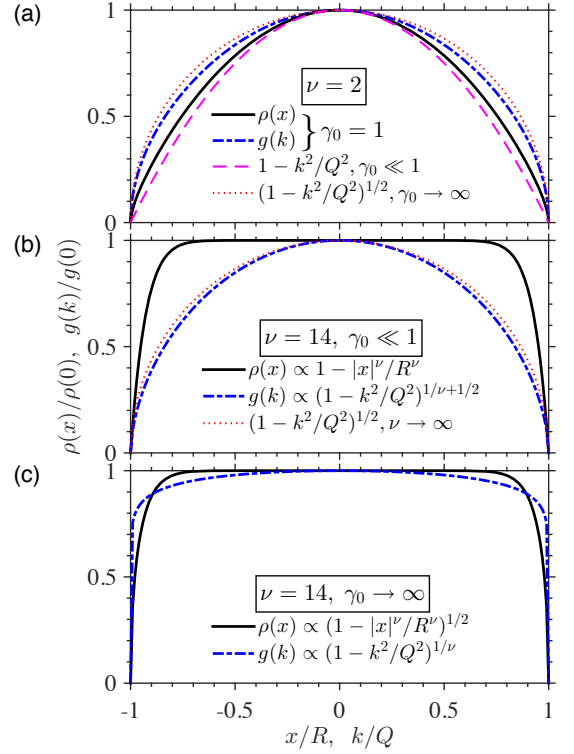


FIG. 2. (Color online) Examples of the initial density profile  $\rho(x)$  (black solid lines) and the quasimomentum distribution  $g(k)$  (blue dash-dotted lines), which determines the shape of the asymptotic density profile  $\rho_\infty(x, t)$  and the momentum distribution  $n_\infty(k, t)$ , Eqs. (5) and (7), with  $Q$  being the maximum (quasi)momentum. (a) Expansion from a harmonic trap, with the solid and dash-dotted lines corresponding to the numerical results for  $\gamma_0 = 1$ . The analytic results (coinciding with the hydrodynamic self-similar solutions) in the weakly ( $\gamma_0 \ll 1$ ) and strongly ( $\gamma_0 \rightarrow \infty$ ) interacting regimes are shown, respectively, by the dashed-magenta and dotted-red lines. The numerical results in these regimes, obtained for  $\gamma_0 = 2.5 \times 10^{-4}$  and  $\gamma_0 = 200$ , are indistinguishable from the respective analytic curves and are omitted from the graphs for clarity. (b) Main curves are for a highly anharmonic trap with  $\nu = 14$  [27] and  $\gamma_0 \ll 1$ ; the semicircle (dashed red line) corresponds to the limiting behavior of  $g(k)$  for a box potential ( $\nu \rightarrow \infty$ ). (c) Strongly interacting regime, for the same  $\nu = 14$ .

where we have introduced a dimensionless parameter  $\lambda(t) = t/t_e = 2c_0 t/R$ , with  $c_0 = \sqrt{\mu_0/m}$  being the sound velocity in the trap center. By comparing this result with the initial density distribution, Eq. (12), it is now easy to see that for  $\nu = 2$ , for which  $I_2 = \pi/4$  and  $t_e = 1/(\sqrt{2}\omega_0)$ , we immediately reproduce the scaling solution of Refs. [15, 16], in which  $\lambda(t) = \sqrt{2}\omega_0 t$  takes the meaning of the single scaling parameter. In the hydrodynamic approach,  $\lambda(t)$  is obtained from the scaling equation  $\dot{\lambda} = \omega_0^2/\lambda^2$  [20]. We can also immediately conclude that such a self-similar scaling solution is not supported by any other power-law trap potential. In particular, the case of  $\nu = 14$  illustrated in Fig. 2 (b) shows a dramatic difference between the density profiles of the initial and expanded clouds (*cf.* [28]).

In the TG regime ( $\gamma_0 \rightarrow \infty$ ), the local semiclassical distribution  $f(k, x; 0)$  is given by [14]

$$f(k, x; 0) = 1/2\pi, \text{ for } |k| < \pi\rho(x), \quad (15)$$

and  $f(k, x; 0) = 0$  otherwise, where  $\pi\rho(x)$  is the maximum quasimomentum coinciding with the Fermi momentum of an ideal uniform Fermi gas of density  $\rho(x)$ .

The density profile  $\rho(x)$  is found from the equation of state of a uniform system  $\rho = \sqrt{2m\mu/(\hbar\pi)^2}$ , yielding

$$\rho(x) = \sqrt{2m\mu(x)/(\hbar\pi)^2} = \rho_0 \sqrt{1 - |x|^\nu/R^\nu}, \quad (16)$$

for  $|x| < R$ , and  $\rho(x) = 0$  otherwise. Here,  $R = (2\mu_0/\alpha_\nu)^{1/\nu}$  is the Thomas-Fermi radius and  $\rho_0 = [2m\mu_0/(\hbar^2\pi^2)]^{1/2} = [m\alpha_\nu N^\nu/(2^\nu I_\nu^\nu \hbar^2 \pi^2)]^{1/(2+\nu)}$  is the peak density, with  $I_\nu$  being the same numerical coefficient as in Eq. (13).

Integrating  $f(k, x; 0)$ , Eq. (15), over position gives

$$g(k) = \frac{R}{\pi} \left( 1 - \frac{k^2}{\pi^2 \rho_0^2} \right)^{1/\nu}, \quad (17)$$

for  $|k| < \pi\rho_0$ , and  $g(k) = 0$  otherwise. The asymptotic density distribution is therefore given by

$$\rho_\infty(x, t) = \frac{\rho_0}{\lambda(t)} \left( 1 - \frac{x^2}{\lambda(t)^2 R^2} \right)^{1/\nu}, \quad (18)$$

where  $\lambda(t) = t/t_e = c_0 t/R$  and  $c_0 = \sqrt{2\mu_0/m}$  is the sound (Fermi) velocity in the trap center. By comparing  $\rho_\infty(x, t)$  with the initial density distribution, Eq. (16), we immediately see that, for finite  $\nu$  (see also [29]), a self-similar scaling solution is again supported only by a quadratic potential, in which case  $\lambda(t) = \omega_0 t$ . In the hydrodynamic approach, this asymptotic behavior is obtained from the scaling equation  $\ddot{\lambda} = \omega_0^2/\lambda^3$  [20] (see also [30]), which also follows from the exact treatment of Ref. [31].

Considering now the coherence properties of an expanding 1D Bose gas, we note that the only length scale entering into the asymptotic momentum distribution  $n_\infty(k, t)$  [through Eqs. (7) and either (13) or (17)] and hence into the respective one-body density matrix  $G^{(1)}(x, x'; t)$  is the *microscopic* correlation length  $\xi_0 = \hbar/\sqrt{m\mu_0}$  [25, 32], corresponding to the healing length  $\xi_0 = \hbar/\sqrt{mg\rho_0}$  for the weakly interacting gas and the mean interparticle separation  $\xi_0 = 1/\rho_0$  for the TG gas. In all cases,  $\xi_0$  is much smaller than the size of the sample  $R$ , which implies complete loss of phase coherence (if there was any initially) typical of fermions and can be viewed as a manifestation of “dynamical fermionization” discussed in Refs. [31, 33]. Such a loss of phase (or first-order) coherence with expansion, which we note does not follow from the hydrodynamic approach, is indeed the case for a weakly interacting gas, for which the initial (zero-temperature equilibrium) coherence length  $l_\phi^{(0)} \sim \xi_0 e^{2\pi/\sqrt{\gamma_0}}$  [34] is exponentially large and can typically be much larger than  $R$ .

Another manifestation of dynamical fermionization during expansion can be seen in the asymptotic behavior of the same-point two-body correlation function  $g^{(2)}(x, x; t)$ : it acquires

[20] a scaling  $\propto 1/t^2$ , implying suppressed correlation in the long time limit. Such a suppression indicates dynamical approach to the fermionized TG regime, where  $g^{(2)}(x, x) = 0$  due to an effective Pauli exclusion [35, 36]. Moreover, our dynamical result can be written as  $g^{(2)}(x, x; t) \propto 1/\gamma(x, t)^2$  using the inverse scaling of the instantaneous interaction constant  $\gamma(x, t) \equiv mg/\hbar^2 \rho_\infty(x, t)$  with density  $\rho_\infty(x, t) \propto 1/t$ . Such a scaling of the  $g^{(2)}$ -function with  $\gamma$  is indeed typical of an equilibrium TG gas [35, 36]. It must be noted though that this result is by no means an indication of equilibration during expansion as the time scale that establishes the  $1/\gamma(x, t)^2$  scaling is still given by  $t_e$  independently of the initial interaction strength; it is true for even an initially weakly interacting gas with  $\gamma(x, 0) \ll 1$  and can emerge long before the instantaneous value of the interaction strength itself becomes “fermionic”,  $\gamma(x, t) \gg 1$ , due to its own scaling of  $\gamma(x, t) \propto t$ .

In summary, we have analyzed the far-from-equilibrium dynamics of the Lieb-Liniger gas in a quantum explosion scenario of a sudden expansion from the confining trap potential. Considering a general class of power-law traps, we have found the asymptotic density profiles and the momentum distributions of the expanding clouds using the stationary phase and local density approximations. The expansion is generally not self-similar, except for the strongly and weakly interacting gases released from a quadratic trap for which our results are in agreement with the known hydrodynamic scaling solutions. In all cases, the expanding clouds lose their phase coherence and display fermionic density fluctuations (not accounted for by the hydrodynamic theory) on a time scale  $t_e \sim R/c_0$  by which the expansion becomes ballistic. The zero-temperature results presented here are qualitatively valid for  $k_B T \ll \mu_0$ , however, our approach can be easily generalized to a nonzero-temperature initial state using the Yang-Yang approach [37], in which case the role of the initial local quasimomentum distribution  $f(k, x; t = 0)$  will be taken by its temperature-dependent counterpart to be found as in Refs. [25, 36] from the solutions to the Yang-Yang integral equations.

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- [1] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
  - [2] A. Lamacraft and J. Moore, in *Ultracold Bosonic and Fermionic Gases*, edited by A. Fetter, K. Levin, and D. Stamper-Kurn (Elsevier, Amsterdam, 2012), Chap. 7, pp. 177–202.
  - [3] T. Kinoshita, T. Wenger, and D. S. Weiss, Nature (London) **440**, 900 (2006).
  - [4] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm, and J. Schmiedmayer, Nature (London) **449**, 324 (2007).
  - [5] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. **98**, 050405 (2007); M. Rigol, V. Dunjko, and M. Olshanii, Nature (London) **452**, 854 (2008).
  - [6] D. M. Gangardt and M. Pustilnik, Phys. Rev. A **77**, 041604

- (2008).
- [7] D. Muth, B. Schmidt, and M. Fleischhauer, *New Journal of Physics* **12**, 083065 (2010).
  - [8] J.-S. Caux and R. M. Konik, *Phys. Rev. Lett.* **109**, 175301 (2012); J.-S. Caux and F. H. L. Essler, *Phys. Rev. Lett.* **110**, 257203 (2013); M. Kormos, A. Shashi, Y.-Z. Chou, J.-S. Caux, and A. Imambekov, *Phys. Rev. B* **88**, 205131 (2013).
  - [9] C. J. Bolech, F. Heidrich-Meisner, S. Langer, I. P. McCulloch, G. Orso, and M. Rigol, *Phys. Rev. Lett.* **109**, 110602 (2012).
  - [10] J. P. Ronzheimer, M. Schreiber, S. Braun, S. S. Hodgman, S. Langer, I. P. McCulloch, F. Heidrich-Meisner, I. Bloch, and U. Schneider, *Phys. Rev. Lett.* **110**, 205301 (2013); L. Vidmar, S. Langer, I. P. McCulloch, U. Schneider, U. Schollwöck, and F. Heidrich-Meisner, *Phys. Rev. B* **88**, 235117 (2013).
  - [11] T. M. Wright, M. Rigol, M. J. Davis, and K. V. Kheruntsyan, *Phys. Rev. Lett.* **113**, 050601 (2014).
  - [12] M. A. Cazalilla and M. Rigol, *New J. Phys.* **12**, 055006 (2010).
  - [13] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, *Rev. Mod. Phys.* **83**, 863 (2011).
  - [14] E. H. Lieb and W. Liniger, *Phys. Rev.* **130**, 1605 (1963).
  - [15] Y. Castin and R. Dum, *Phys. Rev. Lett.* **77**, 5315 (1996).
  - [16] Y. Kagan, E. L. Surkov, and G. V. Shlyapnikov, *Phys. Rev. A* **54**, R1753 (1996); *Phys. Rev. A* **55**, R18 (1997).
  - [17] C. Menotti and S. Stringari, *Phys. Rev. A* **66**, 043610 (2002).
  - [18] P. Pedri, L. Santos, P. Öhberg, and S. Stringari, *Phys. Rev. A* **68**, 043601 (2003).
  - [19] B. Fang, G. Carleo, A. Johnson, and I. Bouchoule, *Phys. Rev. Lett.* **113**, 035301 (2014).
  - [20] See the Supplemental Material at <http://link.aps.org/supplemental/XXX/xxx> for details of the expansion of the many-body wave function, the calculation of the two-body correlation function  $g^{(2)}(x, x; t)$ , and the comparison with the results from the hydrodynamic approach.
  - [21] For 1D Bose gases confined to highly anisotropic cylindrically symmetric trapping potentials the 1D interaction strength is given by  $g \approx 2\hbar\omega_{\perp}a$  [38], where  $a$  is the 3D  $s$ -wave scattering length and  $\omega_{\perp}$  is the frequency of transverse confinement assumed harmonic.
  - [22] D. Jukić, R. Pezer, T. Gasenzer, and H. Buljan, *Phys. Rev. A* **78**, 053602 (2008).
  - [23] For systems with few particles, the problem can be solved numerically and has been considered previously for  $N = 3$  [22], for the case of expansion from a quadratic trap.
  - [24] This procedure is similar to finding the true momentum distribution of a trapped gas, used, e.g., in [39].
  - [25] K. V. Kheruntsyan, D. M. Gangardt, P. D. Drummond, and G. V. Shlyapnikov, *Phys. Rev. A* **71**, 053615 (2005).
  - [26] A. H. van Amerongen, J. J. P. van Es, P. Wicke, K. V. Kheruntsyan, and N. J. van Druten, *Phys. Rev. Lett.* **100**, 090402 (2008); J. Armijo, T. Jacqmin, K. Kheruntsyan, and I. Bouchoule, *Phys. Rev. A* **83**, 021605 (2011); T. Jacqmin, J. Armijo, T. Berrada, K. V. Kheruntsyan, and I. Bouchoule, *Phys. Rev. Lett.* **106**, 230405 (2011).
  - [27] An early realization of a 1D Bose gas in a boxlike potential was reported in Ref. [40]. More recently, a well-characterized power law trap with  $\nu = 13 \pm 2$  was realized in Ref. [28].
  - [28] A. L. Gaunt, T. F. Schmidutz, I. Gotlibovych, R. P. Smith, and Z. Hadzibabic, *Phys. Rev. Lett.* **110**, 200406 (2013).
  - [29] For  $\nu \rightarrow \infty$ , i.e. a boxlike potential, the expansion of a TG gas in our approach is also self-similar, while the deviations from self-similarity discussed in Ref. [41] are due to finite-size effects.
  - [30] E. B. Kolomeisky, T. J. Newman, J. P. Straley, and X. Qi, *Phys. Rev. Lett.* **86**, 4709 (2001).
  - [31] A. Minguzzi and D. M. Gangardt, *Phys. Rev. Lett.* **94**, 240404 (2005).
  - [32] A. G. Sykes, D. M. Gangardt, M. J. Davis, K. Viering, M. G. Raizen, and K. V. Kheruntsyan, *Phys. Rev. Lett.* **100**, 160406 (2008).
  - [33] M. Rigol and A. Muramatsu, *Phys. Rev. Lett.* **94**, 240403 (2005); H. Buljan, R. Pezer, and T. Gasenzer, *Phys. Rev. Lett.* **100**, 080406 (2008).
  - [34] D. S. Petrov, G. V. Shlyapnikov, and J. T. M. Walraven, *Phys. Rev. Lett.* **85**, 3745 (2000).
  - [35] D. M. Gangardt and G. V. Shlyapnikov, *Phys. Rev. Lett.* **90**, 010401 (2003).
  - [36] K. V. Kheruntsyan, D. M. Gangardt, P. D. Drummond, and G. V. Shlyapnikov, *Phys. Rev. Lett.* **91**, 040403 (2003).
  - [37] C. N. Yang and C. P. Yang, *J. Math. Phys.* **10**, 1115 (1969).
  - [38] M. Olshanii, *Phys. Rev. Lett.* **81**, 938 (1998).
  - [39] L. Viverit, S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Phys. Rev. A* **69**, 013607 (2004); M. J. Davis, P. B. Blakie, A. H. van Amerongen, N. J. van Druten, and K. V. Kheruntsyan, *Phys. Rev. A* **85**, 031604 (2012).
  - [40] T. Meyrath, F. Schreck, J. Hanssen, C.-S. Chuu, and M. Raizen, *Phys. Rev. A* **71**, 041604 (2005).
  - [41] A. del Campo and J. G. Muga, *EPL (Europhysics Letters)* **74**, 965 (2006).

# Supplemental Material:

## Sudden expansion of a one-dimensional Bose gas from power-law traps

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### I. THE MANY-BODY WAVE FUNCTION

The Hamiltonian time evolution of the  $N$ -particle wave function  $\Psi(x_1, x_2, \dots, x_N; t)$  describing the dynamics of the Lieb-Liniger gas in free space, after its sudden release at time  $t = 0$  from the confining trap potential, can be written down using an expansion in eigenfunctions  $\Psi_{\{k_j\}}(x_1, x_2, \dots, x_N)$  of the uniform Lieb-Liniger model,

$$\Psi(x_1, x_2, \dots, x_N; t) = \frac{1}{\sqrt{N!}} \int dk_1 \dots dk_N \times b(k_1, \dots, k_N) \Psi_{\{k_j\}}(x_1, x_2, \dots, x_N) e^{-iE_{\{k_j\}}t}. \quad (\text{S1})$$

Here, the eigenfunctions  $\Psi_{\{k_j\}}(x_1, \dots, x_N)$  are characterized by  $N$  different quasimomenta  $\{k_j\} = k_1, k_2, \dots, k_N$  with eigenenergies  $E_{\{k_j\}} = \sum_{j=1}^N \hbar^2 k_j^2 / 2m$ , whereas the expansion coefficients  $b(k_1, \dots, k_N)$  depend on the initial trapping potential and are normalized to  $\int dk_1 \dots dk_N |b(k_1, \dots, k_N)|^2 = 1$ . The eigenfunctions  $\Psi_{\{k_j\}}(x_1, \dots, x_N)$  in the fundamental region  $x_1 < x_2 < \dots < x_N$  can be written down in explicit form as a sum of  $N!$  partial waves [1],

$$\Psi_{\{k_j\}}(x_1, \dots, x_N) = \frac{1}{\sqrt{(2\pi)^N N!}} \times \sum_P (-1)^P e^{i\Theta(k_{P_1}, \dots, k_{P_N})} e^{i \sum_j k_{P_j} x_j}, \quad (\text{S2})$$

enumerated by permutations  $P$ . Each of the  $N!$  terms in the sum is multiplied by the sign of the permutation defined to be  $(-1)^P = \pm 1$  for an even/odd permutation, *i.e.*, a permutation which can be obtained from the identity permutation  $\{k_j\} = k_1, k_2, \dots, k_N$  by an even/odd number of transpositions of adjacent elements. The phase

$$\begin{aligned} \Theta(k_1, \dots, k_N) &= \sum_{j < l} \frac{1}{2i} \log \frac{1 + i \frac{\hbar^2 (k_l - k_j)}{mg}}{1 - i \frac{\hbar^2 (k_l - k_j)}{mg}} \\ &= \sum_{j < l} \tan^{-1} \left[ \frac{\hbar^2 (k_l - k_j)}{mg} \right] \end{aligned} \quad (\text{S3})$$

arises from two-body scattering by the diffractionless delta-function potential of strength  $g$  (see main text).

The structure of the eigenstates in Eq. (S2) has the following physical interpretation: in the fundamental region  $x_1 < x_2 < \dots < x_N$ , choosing the decreasing arrangement of quasimomenta  $k_1 > k_2 > \dots > k_N$  in Eq. (S2) implies that the identity permutation term, in which  $P_j = j$  ( $j = 1, 2, \dots, N$ ),

corresponds to  $N$  particles entering the collision, with the faster particles being behind the slower particles in consecutive order from left to right. The term corresponding to the permutation obtained from the identity by swapping the indices  $i$  and  $j$  describes the scattering of particles with quasimomenta  $k_i$  and  $k_j$  and therefore acquires a scattering phase  $\theta(k_i - k_j) = 2 \tan^{-1}[\hbar^2 (k_i - k_j)/mg]$ . Proceeding in a similar fashion one generates all other terms in the expansion (S2); in particular, the maximally crossed permutation term with  $P_j = N - j + 1$  corresponds to  $N$  particles leaving the collision, with the faster particles having overtaken the slower ones.

Rearranging the quasimomenta  $\{k_Q\} = k_{Q_1}, k_{Q_2}, \dots, k_{Q_N}$  does not change the physical situation; however, the wavefunction (S2) acquires the factor  $(-1)^Q$ . Therefore, in order to avoid double counting of the physical states in the  $N$ -particle bosonic wavepacket (S1), while at the same time ensuring that it remains symmetric, one requires that the coefficients  $b(k_1, \dots, k_N)$  are antisymmetric functions of quasimomenta.

Substituting Eq. (S2) into Eq. (S1) and using the antisymmetry property of the expansion coefficients  $b(k_1, \dots, k_N)$  gives a particularly simple expression for the initial state of the trapped Lieb-Liniger gas,

$$\begin{aligned} \Psi(x_1, x_2, \dots, x_N; 0) &= \frac{1}{(2\pi)^{N/2}} \int dk_1 \dots dk_N \\ &\times b(k_1, \dots, k_N) e^{i\Theta(k_1, \dots, k_N)} e^{i \sum_j k_j x_j}, \end{aligned} \quad (\text{S4})$$

valid in the fundamental region  $x_1 < x_2 < \dots < x_N$ . In any other region the symmetry of the bosonic functions under permutation of particles should be used. Equation (S4) describes the initial  $N$ -body wavepacket; its time evolution after a sudden removal of the confining potential is governed by the free Lieb-Liniger Hamiltonian and is given by

$$\begin{aligned} \Psi(x_1, x_2, \dots, x_N; t) &= \frac{1}{(2\pi)^{N/2}} \int dk_1 \dots dk_N \\ &\times b(k_1, \dots, k_N) e^{i\Theta(k_1, \dots, k_N)} e^{i \sum_j (k_j x_j - \hbar k_j^2 t / 2m)}, \end{aligned} \quad (\text{S5})$$

which is the same as Eq. (1) of the main text.

To evaluate the integral in Eq. (S5) using the stationary phase approximation, we note that the main contribution to the integral comes from the stationary phase points  $k_j^* = mx_j / \hbar t$  corresponding to  $d\phi(k_j)/dk_j = 0$ , where  $\phi(k_j) \equiv kx_j - \hbar k_j^2 t / 2m$ . The integral is then calculated by approximating the slowly varying function  $b(k_1, \dots, k_N) e^{i\Theta(k_1, \dots, k_2)}$  by its value at the stationary points and pulling it outside the integral; the remaining integral over the exponentials of phases  $\phi(k_j)$  are evaluated by using the Taylor expansion near

the points  $k_j^*$  and keeping only the first two nonzero terms,  $\phi(k_j) \approx \phi(k_j^*) + \frac{1}{2}\phi''(k_j^*)(k_j - k_j^*)^2$ . As a result, we obtain

$$\Psi_\infty(x_1, x_2, \dots, x_N; t) = \left(\frac{m}{\hbar t}\right)^{N/2} b\left(\frac{mx_1}{\hbar t}, \dots, \frac{mx_N}{\hbar t}\right) e^{i\Theta\left(\frac{mx_1}{\hbar t}, \dots, \frac{mx_N}{\hbar t}\right) + i\frac{m}{2\hbar t} \sum_j x_j^2} e^{-i\pi N/4}, \quad (\text{S6})$$

which coincides with Eq. (4) of the main text.

## II. TWO-BODY CORRELATION FUNCTION

The calculation of the two-body local (same point) correlation function,

$$G^{(2)}(x, x; t) = \frac{N(N-1)}{2} \times \int dx_3 \dots dx_N |\Psi_\infty(x, x, x_3, \dots, x_N; t)|^2, \quad (\text{S10})$$

requires the knowledge of the wavefunction at two coinciding points. The asymptotic wavefunction Eq. (S6) must, in this case, be modified to take into account the zero of the prefactor

$$G(k_1, k_2, \dots, k_N) \equiv b(k_1, k_2, \dots, k_N) e^{i\Theta(k_1, k_2, \dots, k_N)} \quad (\text{S11})$$

at  $k_1 = k_2 = mx/\hbar t$ . Expanding this prefactor around the stationary points  $k_j^* = mx_j/\hbar t$  to the second order, one obtains the required asymptotic wavefunction,

$$\Psi_\infty(x, x, x_3, \dots, x_N; t) = \frac{m}{i\hbar t} \left(\frac{m}{\hbar t}\right)^{N/2} e^{-i\pi N/4} \times e^{i\frac{m}{2\hbar t} \sum_j x_j^2} \sum_{j=1}^N \left. \frac{\partial^2 G}{\partial k_j^2} \right|_{k_j=k_j^*}. \quad (\text{S12})$$

This form of the wavefunction manifests the effective ‘fermionization’ via the additional factor  $1/t$  as compared to Eq. (S6). Substituting Eq. (S12) into Eq. (S10) and using a variable change  $y_j = mx_j/\hbar t$  for  $j = 3, \dots, N$  results in the following scaling of the pair correlation function with time:  $G^{(2)}(x, x; t) \propto 1/t^4$ . Normalizing this with the product of densities  $\rho_\infty(x, t)$ , yields

$$g^{(2)}(x, t) = \frac{G^{(2)}(x, x; t)}{\rho_\infty(x, t)^2} \propto \frac{1}{t^2}, \quad (\text{S13})$$

which is the result discussed in the main text, indicating the ‘dynamical fermionization’ of expanding bosons.

## III. COMPARISON WITH THE HYDRODYNAMIC APPROACH

As shown in the main text, the free expansion of the Lieb-Liniger gas from a quadratic ( $\nu = 2$ ) trap leads to self-similar scaling solutions in the limits of very weak and very strong

interactions. Such scaling solutions can be obtained from the hydrodynamic approach [2–5], which we would like to illustrate here for instruction.

In the hydrodynamic approach, the evolution of the density and velocity fields,  $\rho(x, t)$  and  $v(x, t)$ , under the variations of the trap frequency  $\omega(t)$ , is governed by the following coupled equations (see, *e.g.*, [4, 5]),

$$\begin{aligned} \partial_t \rho + \partial_x(\rho v) &= 0, \\ \partial_t v + v \partial_x v &= -\partial_x \left( \frac{1}{2} \omega(t)^2 x^2 \right) - \frac{1}{m\rho} \partial_x P, \end{aligned} \quad (\text{S14})$$

where  $P$  is the pressure.

In the weakly interacting regime ( $\gamma_0 \ll 1$ ), the pressure is given by  $P(x, t) = g\rho(x, t)^2/2$ , which follows from the equation of state  $\mu = g\rho$  and the Gibbs-Duhem thermodynamic relation  $\rho = (\partial P/\partial \mu)_T$ . Using this and substituting the scaling ansatz for  $\rho(x, t)$  from Eq. (15) of the main text for  $\nu = 2$  into the above hydrodynamic equations, yields, for  $\omega(t) = 0$  (*i.e.*, for complete opening of the confining trap), the following equation for the scaling parameter  $\lambda(t)$ :

$$\ddot{\lambda} = \omega_0^2/\lambda^2. \quad (\text{S15})$$

The initial conditions are  $\lambda(0) = 1$  and  $\dot{\lambda}(0) = 0$ . Multiplying Eq. (S15) by  $\dot{\lambda}$  one obtains the integral of motion  $\dot{\lambda}^2 + 2\omega_0^2/\lambda = \text{const} = 2\omega_0^2$ , which gives the asymptotic solution  $\lambda(t) = \sqrt{2}\omega_0 t$  for  $t \rightarrow \infty$ , *i.e.*, the same result as the one discussed in the main text.

In the Tonks-Girardeau regime ( $\gamma_0 \rightarrow \infty$ ), the pressure is given by  $P(x, t) = \hbar^2 \pi^2 \rho(x, t)^3/3m$ , from the equation of state  $\mu = \hbar^2 \pi^2 \rho^2/2m$ . In this case, substituting the scaling ansatz (19) of the main text (for  $\nu = 2$ ) into the hydrodynamic equations with  $\omega(t) = 0$  gives the following equation for the scaling parameter:

$$\ddot{\lambda} = \omega_0^2/\lambda^3. \quad (\text{S16})$$

The integral of motion is now  $\dot{\lambda}^2 + \omega_0^2/\lambda^2 = \text{const} = \omega_0^2$ , which gives  $\lambda(t) = \omega_0 t$  for  $t \rightarrow \infty$ , in agreement with what we obtained in the main text. This result can also be obtained [6] within the exact treatment of the Tonks-Girardeau gas using time-dependent Bose-Fermi mapping and the scaling transformation applicable to the time-dependent Schrödinger equation for the single-particle orbitals.

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- [1] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).
  - [2] Y. Castin and R. Dum, Phys. Rev. Lett. **77**, 5315 (1996).
  - [3] Y. Kagan, E. L. Surkov, and G. V. Shlyapnikov, Phys. Rev. A **54**, R1753 (1996); **55**, R18 (1997).
  - [4] C. Menotti and S. Stringari, Phys. Rev. A **66**, 043610 (2002).
  - [5] P. Pedri, L. Santos, P. Öhberg, and S. Stringari, Phys. Rev. A **68**, 043601 (2003).
  - [6] A. Minguzzi and D. M. Gangardt, Phys. Rev. Lett. **94**, 240404 (2005).